

A note on quasitriangularity and trace-class selfcommutators

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In [3] C. A. BERGER and B. I. SHAW proved that for a hyponormal operator T the following inequality holds:

$$\mathrm{Tr}[T^*, T] \leq \frac{1}{\pi} m(T) \omega(\sigma(T))$$

where $\sigma(T)$ is the spectrum of T , ω is planar Lebesgue measure and $m(T) \in \mathbb{N} \cup \{\infty\}$ denotes the multicyclicity of T . The aim of the present note is to give a new proof and an extension of the result of Berger and Shaw by connecting it with quasitriangularity relative to the Hilbert—Schmidt class. Thus, we obtain that the hyponormality condition can be replaced by the condition that the negative part $([T^*, T])_-$ of $[T^*, T]$ be trace class (the author has learned that this result has been obtained about a year ago by C. A. Berger using different methods). But even more, for such T we prove that

$$\mathrm{Tr}[T^*, T] \leq \frac{1}{\pi} m(T+X) \omega(\sigma(T+X))$$

where X is any Hilbert—Schmidt operator. In particular if

$$\mathrm{Tr}[T^*, T] > \frac{1}{\pi} \omega(\sigma(T))$$

then every Hilbert—Schmidt perturbation of T has a non-trivial invariant subspace.

Quasitriangular operators were introduced by P. R. HALMOS [6] and it was shown by APOSTOL, FOIAȘ and VOICULESCU [2] that there is a spectral characterization of these operators. A refinement of the notion of quasitriangular operator relative to a norm-ideal was considered in [11].

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Throughout, H will denote a complex separable Hilbert space of infinite dimension. By $\mathcal{L}(H)$ we denote the bounded operators on H and by $\mathcal{P}(H)$ the set of finite-rank orthogonal projections on H with its natural order. Then the analogue of Apostol's modulus of quasitriangularity relative to a Schatten—von Neumann class is:

$$q_p(T) = \liminf_{P \in \mathcal{P}(H)} |(I-P)TP|_p$$

where $T \in \mathcal{L}(H)$ and $|X|_p = \text{Tr}((X^*X)^{p/2})$ ($1 \leq p < \infty$).

Then if $P_n \in \mathcal{P}(H)$ and $P_n \xrightarrow{w} I$ we have

$$\liminf_{n \rightarrow \infty} |(I-P_n)TP_n|_p \cong q_p(T).$$

Moreover, one can find $P_n \in \mathcal{P}(H)$ such that $P_n \uparrow I$ and

$$\lim_{n \rightarrow \infty} |(I-P_n)TP_n|_p = q_p(T).$$

For $T \in \mathcal{L}(H)$ we shall denote by $\text{Rat}(T)$ the algebra of operators of the form $f(T)$ where f is a rational function with poles off the spectrum $\sigma(T)$ of T . The multicyclicity $m(T) \in \mathbb{N} \cup \{\infty\}$ is the least cardinal of a set $\Xi \subset H$ such that the closed linear span of $\text{Rat}(T)\Xi$ is H .

Proposition 1. For $T \in \mathcal{L}(H)$ and $1 \leq p < \infty$ we have

$$q_p(T) \cong (m(T))^{1/p} \|T\|.$$

Proof. If $m(T) = \infty$ there is nothing to prove. So assume $m(T) = n < \infty$ and consider a multicyclic set $\{\xi_1, \dots, \xi_n\}$ for T . Consider

$$H_j = \bigvee_{k=1}^j \text{Rat}(T)\xi_k, \quad H_0 = 0, \quad K_j = H_j \ominus H_{j-1}, \quad T_j = P_{K_j} T|_{K_j}, \quad \eta_j = P_{K_j} \xi_j.$$

Then, using Proposition 2.1 of [11] we have $q_p(T) \leq \left(\sum_{k=1}^n (q_p(T_k))^p \right)^{1/p}$. Now, it is easily seen that $\sigma(T_k) \subset \sigma(T)$ and η_k is a multicyclic vector for T_k . This reduces the proof of the proposition to the case $n=1$.

Consider a sequence $\{\lambda_j\}_{j=1}^\infty$ of points contained and dense in the union of the bounded components of $\mathbb{C} \setminus \sigma(T)$. Since ξ_1 is multicyclic for T , it is easily seen that denoting by P_m the projection onto the finite-dimensional subspace of H spanned by the vectors $T^k(T-\lambda_1)^{-1} \dots (T-\lambda_m)^{-1} \xi_1$ where $0 \leq k \leq 2m$, we have $P_m \leq P_{m+1}$, $P_m \uparrow I$ and $\text{rank}((I-P_m)TP_m) = 1$. It follows that $|(I-P_m)TP_m|_p \leq \|T\|$ and hence $q_p(T) \leq \|T\|$. O.E.D.

For a hermitian operator $A \in \mathcal{L}(H)$ such that the negative part A_- of A is trace-class, we shall denote by $\text{Tr } A$ the trace of A , in case A is trace-class and ∞ in case A is not trace-class.

Proposition 2. *Let $T \in \mathcal{L}(H)$ be an operator such that the negative part $([T^*, T])_-$ of $[T^*, T]$ is trace-class. Then we have $\text{Tr}[T^*, T] \leq (q_2(T))^2$.*

Proof. Let $P_m \in \mathcal{P}(H)$, $P_m \uparrow I$ be such that $\lim_{m \rightarrow \infty} \|(I - P_m)TP_m\|_2 = q_2(T)$.

We have

$$\begin{aligned} (q_2(T))^2 &= \lim_{m \rightarrow \infty} \|(I - P_m)TP_m\|_2^2 = \lim_{m \rightarrow \infty} \text{Tr}(P_m T^* TP_m - P_m T^* P_m TP_m) = \\ &= \lim_{m \rightarrow \infty} \text{Tr}(P_m [T^*, T] P_m + P_m T(I - P_m)T^* P_m) = \\ &\cong \limsup_{m \rightarrow \infty} \text{Tr}(P_m [T^*, T] P_m) = \text{Tr}[T^*, T]. \end{aligned}$$

Q.E.D.

Proposition 3. *Let $T \in \mathcal{L}(H)$ be an operator such that the negative part $([T^*, T])_-$ of $[T^*, T]$ be trace-class and let $X \in \mathcal{L}(H)$ be a Hilbert—Schmidt operator. Then we have*

$$\text{Tr}[T^*, T] \leq \frac{1}{\pi} m(T+X) \omega(\sigma(T+X))$$

where ω denotes planar Lebesgue-measure.

Proof. It is clearly sufficient to consider the case when $m(T+X) = n < \infty$. Given $\varepsilon > 0$ and denoting by Ω the open set

$$\Omega = \{z \in \mathbb{C}: |z| \leq \|T+X\| + \varepsilon\} \setminus \sigma(T+X)$$

we can find a hyponormal operator D such that:

$$\sigma(D) \subset \Omega, \quad m(D) = n, \quad \|D\| \leq \|T+X\| + \varepsilon, \quad [D^*, D] \geq 0,$$

$$\text{Tr}[D^*, D] \leq \frac{n}{\pi} (\omega(\Omega) - \varepsilon).$$

Such a D is easily constructed by using the “computational lemma” of the paper of BERGER and SHAW [3], or more elementarily by considering an appropriate direct sum of operators of the form $\lambda I + \mu S$ where $\lambda, \mu \in \mathbb{C}$ and S is the unilateral shift.

Using Proposition 1 we have

$$\text{Tr}[(T \oplus D)^*, (T \oplus D)] \leq (q_2(T \oplus D))^2,$$

and hence

$$\mathrm{Tr}[T^*, T] + \frac{n}{\pi}(\omega(\Omega) - \varepsilon) \leq (q_2(T \oplus D))^2.$$

But $q_2(T \oplus D) = q_2((T+X) \oplus D)$ since X is Hilbert—Schmidt.

Moreover, $m((T+X) \oplus D) = n = m(T+X)$ and hence, using Proposition 2, we have

$$(q_2((T+X) \oplus D))^2 \leq (m(T+X))(\|T+X\| + \varepsilon)^2 = \frac{1}{\pi} m(T+X)(\omega(\Omega) + \omega(\sigma(T+X))).$$

It follows that $\mathrm{Tr}[T^*, T] \leq \frac{1}{\pi} m(T+X)(\omega(\sigma(T+X)) - \varepsilon)$. Since $\varepsilon > 0$ is arbitrary, we have

$$\mathrm{Tr}[T^*, T] \leq \frac{1}{\pi} m(T+X) \omega(\sigma(T+X))$$

which is the desired result.

Q.E.D.

Consider $\sigma_{le}(T)$, $\sigma_{re}(T)$ the left-essential and the right-essential spectra of T and remark that if $\sigma(T+X)$ in the proposition above is bigger than $\sigma_{le}(T) \cap \sigma_{re}(T)$ then $T+X$ has a non-trivial invariant subspace. This together with Proposition 3 gives the following:

Corollary 1. *If T is an operator with $([T^*, T])_-$ trace class and*

$$\mathrm{Tr}[T^*, T] > \frac{1}{\pi} (\sigma_{le}(T) \cap \sigma_{re}(T))$$

then every operator $T+X$ with X Hilbert—Schmidt has a non-trivial invariant subspace.

Consider also $E(\sigma(T))$ the polynomially convex hull of $\sigma(T)$, i.e., the complement of the unbounded component of $\mathbb{C} \setminus \sigma(T)$ and remark that for X a compact operator $\sigma(T+X) \cap (\mathbb{C} \setminus E(\sigma(T)))$ is an at most countable set and hence $\omega(\sigma(T+X)) \leq \omega(E(\sigma(T)))$. This together with Proposition 3 gives:

Corollary 2. *If T is an operator with $([T^*, T])_-$ trace-class and if*

$$\mathrm{Tr}[T^*, T] > \frac{1}{\pi} \omega(E(\sigma(T)))$$

then $m(T+X) > 1$ for every Hilbert—Schmidt operator X .

References

- [1] C. APOSTOL, Quasitriangularity in Hilbert space, *Indiana Univ. Math. J.*, **22** (1973), 817—825.
- [2] C. APOSTOL, C. FOIAȘ, D. VOICULESCU, Some results on non-quasitriangular operators. II, *Rev. Roum. Math. Pures et Appl.*, **18** (1973); III, *ibidem* 309; IV, *ibidem* 487; VI, *ibidem* 1473.
- [3] C. A. BERGER, B. I. SHAW, Selfcommutators of multicyclic hyponormal operators are always trace class, *Bull. Amer. Math. Soc.*, **79** (1973), 1193—1199.
- [4] R. G. DOUGLAS, C. PEARCY, A note on quasitriangular operators. *Duke Math. J.*, **37** (1970), 177—188.
- [5] R. G. DOUGLAS, C. PEARCY, *Invariant subspaces of non-quasitriangular operators*, Springer Lecture Notes in Math. N° 345, 13—57.
- [6] P. R. HALMOS, Quasitriangular operators, *Acta Sci. Math.*, **29** (1968), 283—293.
- [7] T. KATO, Smooth operators and commutators, *Studia Math.*, **31** (1968), 535—546.
- [8] C. R. PUTNAM, An inequality for the area of hyponormal spectra, *Math. Z.*, **116** (1970), 323—330.
- [9] C. R. PUTNAM, Trace norm inequalities for the measure of hyponormal spectra, *Indiana Univ. Math. J.*, **21** (1971), 775—779.
- [10] C. PEARCY, *Some recent developments in operator theory*, CBMS Regional Conference Series in Mathematics.
- [11] D. VOICULESCU, Some extensions of quasitriangularity, *Rev. Roum. Math. Pures et Appl.*, **18** (1973), 1303—1320.